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Clebsch–Gordan coefficients and Racah coefficients for the SU(2) and SU(1, 1) groups as the discrete analogues of the Pöschl–Teller potential wavefunctions

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Abstract. The Clebsch–Gordan coefficients (CGC) and the Racah coefficients for the SU(2) and SU(1, 1) groups are studied as functions of a discrete variable. It has been shown that CGC for SU(2) and SU(1, 1) groups may be considered to be discrete analogues of wavefunctions for the one-dimensional Schrödinger equation with the Pöschl–Teller potential. Expressions for CGC and 6*j*-symbols of the SU(1, 1) group have been found through the Hahn and Racah polynomials. Consideration is given to the asymptotic properties of eigenvalues and eigenfunctions of the Hamiltonian of an asymmetric top and of the Bargmann–Moshinsky operator Ω.

1. Introduction

The basic quantities of the quantum theory of angular momenta, the Clebsch–Gordon coefficients (CGC) and the Wigner 6*j*-symbols, are frequently used in various problems of theoretical and mathematical physics and group representation theory (it will be sufficient to mention the problems of atomic and nuclear spectroscopy, the SU(2) group representations, and the special functions). Therefore, the properties of these quantities have been rather well studied; numerous mathematical and physical works are devoted to studying their theory (see e.g. Gelfand *et al* 1958, Vilenkin 1965a, Smorodinsky and Shelepin 1972, Varshalovich *et al* 1975, Jucys and Bandzaitis 1977, Sviridov and Smirnov 1977).

Until recently, however, the deep relationships between CGC and Wigner 6*j*-symbols of the SU(2) group on the one hand and the classical orthogonal polynomials of a discrete variable (Hahn 1949, Weber and Erdelyi 1952, Bateman and Erdelyi 1953, Karlin and McGregor 1961, Nikiforov and Uvarov 1978, 1983, Askey and Wilson 1979, Wilson 1980, Nikiforov *et al* 1982) on the other hand passed unnoticed. The CGC of the SU(2) group proved to be expressible through the Hahn polynomials (Gelfand *et al* 1958, Ryvkin 1959, Meckler 1959, Kirichenko and Stepanovsky 1974, Koornwinder 1981, Smorodinsky and Suslov 1982a, Nikiforov and Suslov 1982, Nikiforov *et al* 1983a) which are discrete analogues of the Jacobi polynomials on a grid with a constant step. A similar result has been obtained for the Wigner 6*j*-symbols. They may be identified with the Racah polynomials (Wilson 1980, Nikiforov *et al* 1983a, Smorodinsky and Suslov 1982b, Suslov 1983) which are the discrete analogues of the Jacobi polynomials on a quadratic grid (Nikiforov *et al* 1982). Thus, the vector coupling coefficients and the 6*j*-symbols have found their place as discrete analogues of the Jacobi polynomials in the theory of special functions.

From such a viewpoint, the orthogonality property, the difference equations, the Rodrigues formula, the recurrence relations, the asymptotics, etc are natural. The character of the CGC and $6j$-symbol behaviour as the parameters vary and the nature of the additional selection rules relevant to the roots of these quantities (Smorodinsky and Suslov 1982a, b) become clear. Thereby, the quantum theory of angular momentum gets even more complete and logically consistent.

The present work continues the study of CGC as functions of a discrete variable. In § 2 it is shown that CGC of the SU(2) and SU(1, 1) groups may be considered as discrete analogues of wavefunctions for the one-dimensional Schrödinger equation with the Pöschl–Teller potential. After that, the SU(2) group results (Ryvkin 1959, Meckler 1959, Kirichenko and Stepanovsky 1974, Koornwinder 1981, Smorodinsky and Suslov 1982a, b, Nikiforov and Suslov 1982, Nikiforov *et al* 1983a, b, Suslov 1983) are generalised for the Kronecker product of two unitary irreducible representations belonging to the discrete positive series of the SU(1, 1) group. In §§ 3 and 4 the explicit expressions for CGC and $6j$-symbols of the SU(1, 1) group will be found through the Hahn and Racah polynomials respectively. Asymptotic formulae of the second order of accuracy are obtained for the above-mentioned quantities (analogous formulae for the SU(2) group are presented by Nikiforov *et al* (1983b)). Section 5 deals with the asymptotic properties of the eigenvalues and eigenfunctions of the Hamiltonian of the asymmetric top and of the Bargmann–Moshinsky operator Ω (Bargmann and Moshinsky 1961) in the limit of large angular momentum. The relationships of the generalised spherical functions for the discrete positive series of the SU(1, 1) group to the Meixner polynomials and the standardisation of the Hahn and Racah polynomials are noted in appendices 1 and 2.

2. Clebsch–Gordan coefficients as discrete analogues of the Pöschl–Teller potential wavefunctions

CGC for the SU(2) group obey the standard recurrence relation

$$\begin{aligned}&[(j_1-m_1)(j_1+m_1+1)(j_2+m_2)(j_2-m_2+1)]^{1/2}\langle j_1, m_1+1, j_2, m_2-1|jm\rangle\\&\quad+[(j_1+m_1)(j_1-m_1+1)(j_2-m_2)(j_2+m_2+1)]^{1/2}\langle j_1, m_1-1, j_2, m_2+1|jm\rangle\\&\quad=-[j_1(j_1+1)+j_2(j_2+1)-j(j+1)+2m_1m_2]\langle j_1m_1j_2m_2|jm\rangle\end{aligned}\tag{2.1}$$

which follows from the fact that the function

$$|j_1j_2jm\rangle=\sum_{m_1+m_2=m}\langle j_1m_1j_2m_2|jm\rangle|j_1m_1\rangle|j_2m_2\rangle\tag{2.2}$$

must be the eigenfunction for the operator $J^2=J_-J_++J_0^2+J_0$ $(J_i=J_i(1)+J_i(2))$ with eigenvalue $j(j+1)=-\lambda$.

Let us begin with examining GCG for the SU(2) group. Since $-j_1\leqslant m_1\leqslant j_1$, $-j_2\leqslant m_2\leqslant j_2$, $-j\leqslant m\leqslant j$, after introducing the designations $C_n=(-1)^{j_1-m_1}\langle j_1m_1j_2m_2|jm\rangle$, $n=j_1-m_1$, we may rewrite (2.1) as a second-order difference equation:

$$\begin{aligned}&-C_{n+1}[(n+1)(m+m'+N-n-1)(m-m'+n+1)(N-n-1)]^{1/2}\\&\quad-C_{n-1}[n(m+m'+N-n)(N-n)(m-m'+n)]^{1/2}\\&\quad=C_n[\lambda+\tfrac{1}{4}(N+m+m')^2+\tfrac{1}{4}(N+m-m')^2-\tfrac{1}{2}+\tfrac{1}{2}m^2-\tfrac{1}{2}(N-2n+m'-1)^2].\end{aligned}\tag{2.3}$$

Assume now that $m \geqslant m' = j_1 - j_2 \geqslant 0$, $N = j_1 + j_2 + 1 - m$. Then (2.3) should be solved at the boundary conditions

$$C_{-1} = C_N = 0. \tag{2.4}$$

As was shown by Nikiforov and Suslov (1982), the solutions (2.4) are proportional to the Hahn polynomials which constitute one of the types of the orthogonal polynomials of a discrete variable.

Difference equations are known to have much in common with differential equations of the same order. To have a general idea of the character of the solutions of the difference equation, it is helpful to use the limit $n \gg 1$ when the discrete variable may approximately be considered as continuous, and the difference equation (2.3) may be replaced by an asymptotic differential equation. Consider, therefore, the case $j_1, j_2 \gg m$, m', j (this means that n, $N - n \gg m$, m', j) and produce the expansion of the left-hand side of (2.3) in a power series with respect to $1/n$ and $1/(N-n)$, retaining the terms not above second order. By introducing the variable θ related to n as

$$(\cos\theta)^2 = n/N = (j_1 - m_1)/(j_1 + j_2 - m + 1)$$

we obtain that (2.3) turns, in the above-mentioned approximation, into the following equation for four-dimensional spherical functions (Vilenkin 1965a, Smirnov and Shustov 1981):

$$\frac{1}{\sin\theta\cos\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial C}{\partial\theta}\right) - \left(\frac{(m+m')^2}{(\cos\theta)^2} + \frac{(m-m')^2}{(\sin\theta)^2}\right)C = 4\lambda C \tag{2.5}$$

with boundary conditions $C(\theta = 0) = C(\theta = \pi/2) = 0$; hence it follows that in this extreme case CGC turn into Jacobi polynomials

$$P_{j-m}^{(m+m',m-m')}(\cos 2\theta) = P_{j-m}^{(m+m',m-m')}\left(\frac{m_2 - m_1 + j_1 - j_2 - 1}{j_1 + j_2 - m + 1}\right).$$

This is also in agreement with the limiting form of the Hahn polynomials (Nikiforov and Suslov 1982). The Jacobi polynomials may be related to the Wigner D-function through the replacement $\theta = \beta/2$. As a result we obtain the well known relation

$$\langle j_1 m_1 j_2 m_2 | jm\rangle(-1)^{j_1 - m_1} \approx d^j_{m, j_1 - j_2}(\cos\beta)\ \text{constant} \tag{2.6}$$

(where $\cos\beta = (m_2 - m_1 + j_1 - j_2 - 1)/(j_1 + j_2 - m + 1)$) which makes it possible to treat CGC for the SU(2) group as a discrete analogue for the D-function or four-dimensional spherical harmonic $Y_{LL_1L_2}(\theta)$ of the O(4) group, where $L = 2j$, $L_1 = m + m'$, $L_2 = m - m'$. The tree graphs characterising the coupling scheme of several angular momenta were used by Kuznetsov and Smorodinsky (1975) and Smirnov and Shitikova (1977). Each branching of the tree graph shown in figure 1(a) is in correspondence with the CGC $\langle j_1 m_1 j_2 m_2 | jm\rangle$. On the other hand, the structure of hyperspherical harmonics is also given by a tree graph (Vilenkin 1965a, b, Vilenkin *et al* 1965, Kildyushov 1972, Smirnov and Shustov 1981) containing elements (b), (c) (see figure 1). The above examination shows that the two kinds of tree graphs (the Wigner tree and the hyperspherical tree) are not only alike in their geometric structure but also interrelated closely in essence, namely in the asymptotic extreme j_1, $j_2 \gg m$, m', j the branching of the type of figure 1(a) turns into branching 1(b) with $L_1 = m_1 + m_2 + j_1 - j_2$, $L_2 = m_1 + m_2 - j_1 + j_2$, $L = 2j$. In the particular case $m' = j_1 - j_2 = 0$ the branching 1(a) turns out to be the branching 1(c) (the Jacobi polynomial turns out to be the associated Legendre polynomial of

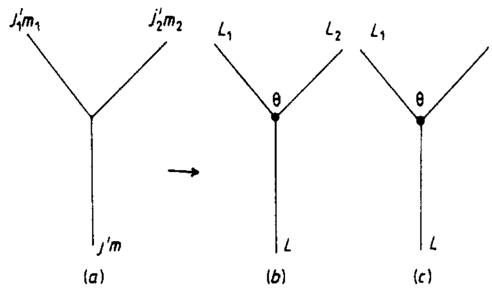

Figure 1.

polar angle $0 \leq \beta \leq \pi$, while $Y_{LL_1L_2}(\theta)$ becomes a standard three-dimensional spherical function $Y_{jm}(\beta)$).

Turning from the function $C(\theta)$ to the function

$$W_{LL_1L_2}(\theta) = (\sin\theta\cos\theta)^{1/2}C(\theta) \approx [n(N-n)/N^2]^{1/4}C_n$$

we reduce (2.5) to the form of the one-dimensional Schrödinger equation

$$-\frac{\partial^2 W}{\partial\theta^2} + \left(\frac{\lambda_1(\lambda_1+1)}{(\sin\theta)^2} + \frac{\lambda_2(\lambda_2+1)}{(\cos\theta)^2} - \varepsilon\right)W = 0 \tag{2.7}$$

where $\lambda_1 = m - m' + \frac{1}{2}$, $\lambda_2 = m + m' + \frac{1}{2}$, $\varepsilon = (2j+1)^2$.

It is seen from the above that the CGC may be treated as a discrete analogue of the wavefunctions for the Pöschl–Teller potential. The spectrum of the levels of this potential is known to be of the form (Flügge 1971)

$$\varepsilon = (\lambda_1 + \lambda_2 + 2k + 2)^2, \qquad k = 0, 1, 2, \ldots. \tag{2.8}$$

This means that j takes the values $j = m$, $m+1, \ldots$. At a fixed m these values are allowed in the angular momentum theory. The difference between the asymptotic and accurate results is that the number of levels in the Pöschl–Teller potential is infinite, whereas the values of j in the CGC $\langle j_1m_1j_2m_2|jm\rangle$ are limited by $m \leq j \leq j_1 + j_2$. This difference is due to the fact that in the region $j \sim j_1 + j_2 = N + m - 1$ the asymptotic conditions are not satisfied, the variable n cannot already be considered continuous, and the asymptotic equation is inapplicable. Therefore, the spectrum of the difference equation (2.3) will contain a finite number of eigenvalues λ, in contrast to the spectrum of (2.7). At small values of j, however, the CGC should behave like the wavefunctions of the Schrödinger equation (2.7). Namely, at $j = m$ the solution is nodeless, i.e. all CGC $(-1)^{j_1-m_1}\langle j_1m_1j_2m_2|jm\rangle$ have the same sign; CGC with $j = m+1$, $m+2, \ldots$ oscillate and change their sign $1, 2, \ldots$ times. These properties of CGC are illustrated in figure 2 which presents the values of the CGC $(-1)^m\langle 10, m, j0|10, m\rangle$ as functions of m (the CGC were taken to be non-normalised; it was assumed that $\langle 10, 10, j0|10, 10\rangle = 1$).

From the figure one can clearly see the oscillating character of the CGC; however, the node of the appropriate wavefunction is not always coincident with an integral or semi-integral value of m. Despite their oscillations, therefore, CGC are rarely vanishing exactly. For example, in the case shown in figure 2 only the CGC $\langle 10, m, 40|10, m\rangle$ have the non-trivial exact node at $m = \pm 9$. From the physical viewpoint, however, the fact that the absolute value of the CGC is small in the region of sign reversal is more important than the exact vanishing of the CGC. This circumstance will result, for

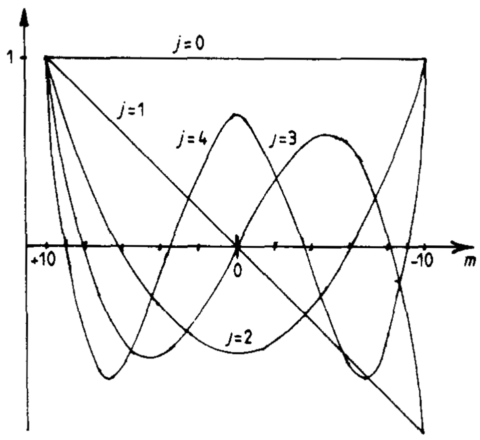

Figure 2. The (non-normalised) Clebsch–Gordan coefficients $(-1)^m\langle 10,\, m,\, j\; 0|10,\, m\rangle$ as functions of m.

example, in peculiar 'beats' of the probability of the transitions of multipolarity $\Lambda = 1, 2, \ldots$ between the levels $|jm\rangle$ of term j ($j \gg 1$) split in an electric, or crystal, or magnetic field. These properties must also be felt in the many-photon coherent transitions in the two-level Dicke-system, etc.

Examine now the direct product of two unitary irreducible representations (UIR) belonging to positive discrete series of the SU(1, 1) group

$$D^{j_1+} \otimes D^{j_2+} = \sum_{j=j_1+j_2+1}^{\infty} \oplus D^{j+}. \tag{2.9}$$

The same recurrence relation as for the SU(2) group is valid for the CGC of the SU(1, 1) group:

$$\begin{aligned}
&[(m_1-j_1)(m_1+j_1+1)(m_2+j_2)(m_2-j_2-1)]^{1/2}(j_1,\, m_1+1,\, j_2,\, m_2-1|jm) \\
&\qquad + [(m_1+j_1)(m_1-j_1-1)(m_2-j_2)(m_2+j_2+1)]^{1/2}(j_1,\, m_1-1,\, j_2,\, m_2+1|jm) \\
&\quad = [j_1(j_1+1)+j_2(j_2+1)-j(j+1)+2m_1m_2](j_1m_1j_2m_2|jm\rangle
\end{aligned} \tag{2.10}$$

but with $m_1 \geqslant j_1+1$, $m_2 \geqslant j_2+1$, $m \geqslant j+1$, $j \geqslant j_1+j_2+1$. By introducing the designations

$$m_1/m = (\cos\theta)^2, \qquad m_2/m = (\sin\theta)^2, \qquad C(\theta) = (j_1m_1j_2m_2|jm),$$

and only retaining in the left-hand side of (2.10) the terms not above second order with respect to $1/m_1$ and $1/m_2$, we reduce (2.10) to the asymptotic form

$$\begin{aligned}
&\frac{1}{\sin\theta\cos\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\cos\theta\frac{\partial C}{\partial\theta}\right) - \left(\frac{(2j_1+1)^2}{(\cos\theta)^2} + \frac{(2j_2+1)^2}{(\sin\theta)^2}\right)C = 4\lambda C, \\
&C(\theta=0) = C(\theta=\pi/2) = 0,
\end{aligned} \tag{2.11}$$

which is nothing other than an equation for the four-dimensional spherical function $Y_{LL_1L_2}(\theta)$ in the case of the O(2, 2) group with $L_1 = 2j_1+1$, $L_2 = 2j_2+1$, $\lambda = -j(j+1)$, $L = 2j$. From this it is seen that CGC of the form $(j_1m_1j_2m_2|jm)$ in the above-mentioned asymptotic limit must reduce to the Jacobi polynomial

$$P^{(2j_1+1,2j_2+1)}_{j-j_1-j_2-1}((m_1-m_2)/m)$$

or to the Wigner D-function

$$d^j_{j_1+j_2+1,j_1-j_2}((m_1-m_2)/m).$$

Through the replacement $W=(\sin\theta\cos\theta)^{1/2}C$, equation (2.11) reduces again to the Pöschl–Teller equation (2.7) with $\lambda_1=2j_2+\frac{1}{2}$, $\lambda_2=2j_1+\frac{1}{2}$ whose energy levels are given by the formula

$$\varepsilon=(\lambda_1+\lambda_2+2k+2)^2=(2j_1+2j_2+2k+3)^2=(2j+1)^2.$$

Hence

$$j=j_1+j_2+1+k, \qquad k=0,1,2,\ldots. \tag{2.12}$$

Thereby, turning to the asymptotic limit m_1, $m_2\to\infty$ at fixed j_1, j_2, m, $m'=m_1-m_2$, we could have found the structure of the Clebsch–Gordan series (2.9), for a Kronecker product of two UIR of positive discrete series, since each level of (2.12) is in correspondence with one of the terms of the series (2.9). It should be noted that, despite the asymptotic character of (2.7), its level spectrum must be in accurate correspondence with the Clebsch–Gordan series, since the structure of this series is independent of the values of m_1, m_2, i.e. is conserved also at high values of these numbers.

The exact spectrum of eigenvalues λ of the difference equation (2.10), like the energy spectrum of the Pöschl–Teller potential, is discrete and *unlimited from above.* This is the difference of the given result from the case of the SU(2) group.

Bearing in mind the graphic technique, the connection between the Wigner tree for the SU(1, 1) group and the hyperspherical tree for hyperboloid harmonics (Smirnov and Shustov 1981) is shown in figure 3. The meaning of the graphs in figure 3 is the same as in figure 1; the branching 3(a) reduces asymptotically to the branching 3(b), $L_1=2j_1+1$, $L_2=2j_2+1$, $L=2j$, $(\cos\theta)^2=m_1/m$; at $j_1=j_2$ the branching 3(a) turns into 3(c).

From the fact that CGC for the SU(2) group and CGC of the type of $D^{j_1+}\otimes D^{j_2+}$ for the SU(1, 1) group satisfy Schrödinger equations of the same type whose potential terms become identical after the replacement

$$\begin{aligned} &J_1=\tfrac{1}{2}(j_1-j_2+m_1+m_2-1), \qquad M_1=\tfrac{1}{2}(j_1+j_2+m_1-m_2+1),\\ &J_2=\tfrac{1}{2}(-j_1+j_2+m_1+m_2-1), \qquad M_2=\tfrac{1}{2}(j_1+j_2-m_1+m_2+1),\\ &J=j, \qquad M=j_1+j_2+1, \end{aligned} \tag{2.13}$$

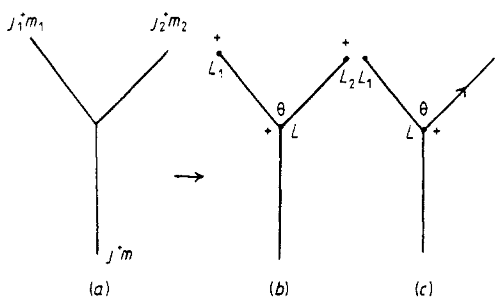

Figure 3.

it follows that relationships may be found between CGC of the SU(2) group and CGC for the positive discrete series of the SU(1, 1) group (within the phase factor)

$$(j_1 m_1 j_2 m_2 | jm) = \langle J_1 M_1 J_2 M_2 | JM \rangle \tag{2.14}$$

where J_i, M_i are related to j_i, m_i through (2.13). This result was obtained by a direct method by Rasmussen (1975) and Chacon *et al* (1975) on the basis of the complementarity of the SU(1, 1) group and the O(4) group in the space of symmetric representation of the Sp(8, R) group.

Examine now CGC for the direct product

$$D^{j_1 +} \otimes D^{j_2 -} = \sum_{j_{min}}^{j_{max}} D^{j+} + \int \mathrm{d}\sigma\, D^{-1/2+\mathrm{i}\sigma} \tag{2.15}$$

which, according to Mukunda and Radhakrishnan (1974), contains the discrete series $D^{j+}, j_{min} \leqslant j \leqslant j_{max}$ (for $j_1 \geqslant j_2$, $j_{max} = j_1 - j_2 - 1$, $j_{min} = 0$ or $\frac{1}{2}$ for integral or half-integral values of $j_1 - j_2$ respectively) and one of the principal series of the UIR $D^j, j = -\frac{1}{2} + \mathrm{i}\sigma$, $0 \leqslant \sigma < \infty$.

The recurrence relation (2.1) is valid for the CGC $(j_1^+ m_1 j_2^- m_2 | jm)$ but now $m_1 \geqslant j_1 + 1$, $m_2 \leqslant -j_2 - 1$, $m = m_1 + m_2$. Assuming that $j_1 \geqslant j_2$, $m_1 \gg j_1$, $|m_2| \gg j_2$, $m_1 - |m_2| \gg j_1, j_2, j$, introducing the designations $(\cosh\theta)^2 = m_1/m$, $(\sinh\theta)^2 = |m_2|/m$, $C(\theta) = (j_1^+ m_1 j_2^- m_2 | jm)$ and retaining only the terms not above second order with respect to $1/m_1$ and $1/|m_2|$ in the left-hand side of (2.1), we reduce the recurrence relation (2.1) to the asymptotic form

$$\frac{1}{\sinh\theta\cosh\theta}\frac{\partial}{\partial\theta}\left(\sinh\theta\cosh\theta\frac{\partial C}{\partial\theta}\right) + \left(\frac{(2j_1+1)^2}{(\cosh\theta)^2} - \frac{(2j_2+1)^2}{(\sinh\theta)^2}\right)C = 4\lambda C. \tag{2.16}$$

Expression (2.16) is an equation for four-dimensional harmonics corresponding to the reduction O(2, 2) ⊃ O(2) × O(2), i.e. to the branching 4(b) in figure 4. After the replacement $W_{LL_1L_2}(\theta) = (\sinh\theta\cosh\theta)^{1/2} C$, (2.16) takes the Schrödinger form

$$-\frac{\partial^2 W}{\partial\theta^2} + \left(\frac{\lambda_1(\lambda_1+1)}{(\sinh\theta)^2} - \frac{\lambda_2(\lambda_2+1)}{(\cosh\theta)^2} - \varepsilon\right)W = 0 \tag{2.17}$$

where $\lambda_1 = 2j_1 + \frac{1}{2}$, $\lambda_2 = 2j_2 + \frac{1}{2}$, $\varepsilon = -(2j+1)^2$, $L_1 = 2j_2 + 1$, $L_2 = 2j_1 + 1$. Obviously, the Schrödinger equation with the potential

$$V(\theta) = \lambda_1(\lambda_1+1)/(\sinh\theta)^2 - \lambda_2(\lambda_2+1)/(\cosh\theta)^2$$

shows both discrete (at $\lambda_2 > \lambda_1$, $j_1 > j_2$) and continuous ($j = -\frac{1}{2} + \mathrm{i}\sigma$, $\varepsilon = \sigma^2 > 0$) spectra of levels. The discrete levels will be in correspondence with the discrete series D^{j+} in

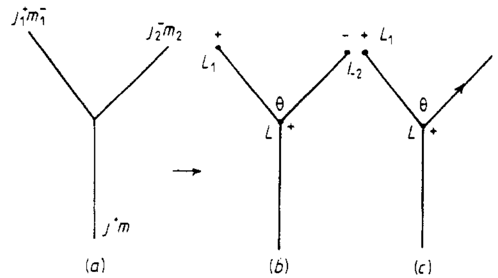

Figure 4.

the expansion (2.15). We shall limit ourselves below to examination of the discrete part of the spectrum. The discrete level spectrum is known (Limic *et al* 1966, 1967, Smirnov and Shustov 1981) to be given by the formula

$$\varepsilon = -(2j+1)^2, \qquad j = j_1 - j_2 - 1 - k, \qquad k = 0, 1, \ldots, [j_1 - j_2 - 1], \tag{2.18}$$

where $[c]$ is the integral part of c.

Thus we can see again that, by studying the level spectrum of the asymptotic Schrödinger equation (2.16), we find the spectrum of the eigenvalues of the difference equation (2.1) exactly and determine the structure of the Clebsch–Gordan series for the UIR of the SU(1, 1) group. Knowing the solutions for (2.16) (Smirnov and Shustov 1981), we may state that the CGC $(j_1^+ m_1 j_2^- m_2 | j^+ m)$ are proportional to the Jacobi polynomials with non-standard indices

$$P_{j-j_1-j_2-1}^{(-(2j_2+1), 2j_1+1)}((m_1 - m_2)/(m_1 + m_2)).$$

These orthogonal polynomials are determined at the interval $(1, +\infty)$ and are featured by the fact that the number of them is finite, i.e. they fail to form a complete system, but all the usual properties of orthogonal polynomials (the Rodrigues formula, the explicit form of the normalisation factor, the properties of roots, etc) are applicable to them.

The same method may be used to analyse CGC for the Kronecker products of UIR of other types. This will be the subject of our subsequent work. Now we shall obtain explicit expressions for CGC of the SU(2) and SU(1, 1) groups through the Hahn polynomials by strictly solving the difference equation (2.1).

3. Clebsch–Gordan coefficients and Hahn polynomials

As was noted above, the CGC of the SU(2) group satisfy the difference equation (2.1). The expression of them through the Hahn polynomials is of the form (Nikiforov and Suslov 1982, Nikiforov *et al* 1983a)

$$\langle j_1 m_1 j_2 m_2 | jm \rangle = \frac{(-1)^{j_1 - m_1}}{d_{j-m}} [\rho(j_2 - m_2)]^{1/2} h_{j-m}^{(m-m', m+m')}(j_2 - m_2, j_1 + j_2 - m + 1)$$
$$(m \geq m' \geq 0,\ m' = j_1 - j_2) \tag{3.1}$$

where $\rho(x)$ and d_n are respectively the weight and norm of the polynomials $h_n^{(\alpha,\beta)}(x, N)$ (see appendix 2). Besides that, the CGC of SU(2) may be expressed through the dual Hahn polynomials $w_n(p) = w_n^{(c)}(p, a, b)$, $p = p(x) = x(x+1)$:

$$\langle j_1 m_1 j_2 m_2 | jm \rangle = (-1)^{j_1 + j_2 - j} \frac{[\rho(j)(2j+1)]^{1/2}}{d_{j_2 - m_2}} w_{j_2 - m_2}^{(m')}[j(j+1); m, j_1 + j_2 + 1]$$
$$(m \geq m' \geq 0,\ m' = j_1 - j_2). \tag{3.2}$$

Here $\rho(x)$ and d_n are respectively the weight and norm of the polynomials $w_n(p)$ (see appendix 2).

Let us obtain now relations analogous to (3.1) and (3.2) for the CGC of the SU(1, 1) group in the case of a Kronecker product of two UIR belonging to the discrete positive series. We shall seek for the corresponding coefficients as solutions of the difference equation (2.10). The substitution

$$(j_1 m_1 j_2 m_2|jm) = [(m_1+j_1)!(m_2+j_2)!/(m_1-j_1-1)!(m_2-j_2-1)!]^{1/2} u_{m_1 m_2}$$

leads to the equation

$$(m_1+j_1+1)(m_2-j_2-1)u_{m_1+1,m_2-1} + (m_1-j_1-1)(m_2+j_2+1)u_{m_1-1,m_2+1}$$
$$= [j_1(j_1+1)+j_2(j_2+1)-j(j+1)+2m_1 m_2]u_{m_1 m_2}. \qquad (3.3)$$

The normalised solution for (3.3) is of the form

$$u_{m_1 m_2} = (1/d_{j-j_1-j_2-1})h^{(2j_1+1,2j_2+1)}_{j-j_1-j_2-1}(m_2-j_2-1;\, m-j_1-j_2-1), \qquad (3.4)$$

where d_n is a norm of the Hahn polynomial $h_n^{(\alpha,\beta)}(x, N)$.

As a result, we find the representation of the CGC of the SU(1, 1) group in the case $D^{j_1^+} \otimes D^{j_2^+}$ through the Hahn polynomials

$$(j_1 m_1 j_2 m_2|jm) = \{[\rho(m_2-j_2-1)]^{1/2}/d_{j-j_1-j_2-1}\}h^{(2j_1+1,2j_2+1)}_{j-j_1-j_2-1}(m_2-j_2-1;\, m-j_1-j_2-1). \qquad (3.5)$$

The weight $\rho(x)$ and norm d_n of the polynomials $h_n^{(\alpha,\beta)}(x, N)$ are presented in appendix 2.

Using the relation between the Hahn polynomials $h_n^{(\alpha,\beta)}(x, N)$ and the dual Hahn polynomials $w_n^{(c)}(p; a, b)$, $p = x(x+1)$ (Karlin and McGregor 1961):

$$h_n^{(\alpha,\beta)}(k, N) = (-1)^{k-n}\frac{k!(N-k-1)!\Gamma(\beta+n+1)}{n!(N-n-1)!\Gamma(\beta+k+1)}$$
$$\times w_k^{((\beta-\alpha)/2)}\left[p\left(\frac{\alpha+\beta}{2}+n\right); \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+N\right] \qquad (k, n = 0, 1, \ldots, N-1), \qquad (3.6)$$

we obtain from (3.5)

$$(j_1 m_1 j_2 m_2|jm) = (-1)^{m_2-j+j_1}\frac{[\rho(j)(2j+1)]^{1/2}}{d_{m_2-j_2-1}} w^{(j_2-j_1)}_{m_2-j_2-1}[j(j+1); j_1+j_2+1, m], \qquad (3.7)$$

where $\rho(x)$ and d_n are respectively the weight and norm of the polynomials $w_n(p)$ (see appendix 2).

The expression of the CGC of the SU(2) and SU(1, 1) groups in terms of the Hahn polynomials makes it possible to study these quantities using the standard methods developed in the theory of orthogonal polynomials. We may indicate, for example, the asymptotic formulae of the second order of accuracy. Using the asymptotics of the Hahn polynomials $h_n^{(\alpha,\beta)}(x, N)$ (Nikiforov *et al* 1983b)

$$h_n^{(\alpha,\beta)}[\tfrac{1}{2}\tilde{N}(1+s) - \tfrac{1}{2}(\beta+1), N] = \tilde{N}^n[P_n^{(\alpha,\beta)}(s) + O(\tilde{N}^{-2})],$$
$$\tilde{N} = N + (\alpha+\beta)/2, \qquad N \to \infty, \qquad (3.8)$$

for the CGC of the SU(2) group, we get the relation (Nikiforov *et al* 1983b)

$$\langle j_1 m_1 j_2 m_2|jm\rangle \approx (-1)^{j_2+m_2}\left(\frac{2j+1}{j_1+j_2+1}\right)^{1/2} d^j_{mm'}(\beta), \qquad \cos\beta = \frac{m_1-m_2}{j_1+j_2+1}, \qquad (3.9)$$

which is valid at $j_1+j_2+1 \gg 1$, $m \approx m' \sim 1$ ($m \geqslant m' \geqslant 0$, $m' = j_1 - j_2$).

Similarly, we obtain from (3.5) and (3.8) for the CGC of the SU(1, 1) group at $m \gg j \sim 1$:

$$(j_1 m_1 j_2 m_2 | jm) \approx (-1)^{2j_1+1} \left(\frac{2j+1}{m}\right)^{1/2} d^j_{j_1+j_2+1, j_2-j_1}(\beta), \qquad \cos\beta = \frac{m_2 - m_1}{m}. \tag{3.10}$$

The functions $d^j_{mm'}(\beta)$ entering (3.9) and (3.10) are determined in the same way as in Varshalovich *et al* (1975).

4. The Wigner 6j-symbols and the Racah polynomials

As was shown by Wilson (1980), Smorodinsky and Suslov (1982b), Nikiforov *et al* (1983a, b) and Suslov (1983) the Wigner 6j-symbols for the SU(2) group may be expressed in terms of the Racah polynomials which are discrete analogues of the Jacobi polynomials on a quadratic grid (Nikiforov *et al* 1982). The formula corresponding to the standardisation adopted here for the Racah polynomials $v_n^{(\alpha,\beta)}(p, a, b)$ is of the form (Nikiforov *et al* 1983a, b)

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \frac{(-1)^{j_1+j+j_{23}}}{(2j_{12}+1)^{1/2}} \times \frac{[\rho(j_{23})]^{1/2}}{d_{j_{12}-j_1+j_2}} v^{(m-m', m+m')}_{j_{12}-j_1+j_2}[j_{23}(j_{23}+1); j_3-j_2, j_2+j_3+1], \tag{4.1}$$

where $\rho(x)$ and d_n are respectively the weight and norm of the polynomials $v_n^{(\alpha,\beta)}(p, a, b)$ (see appendix 2), $m = j_1 - j_2$, $m' = j_3 - j$; $j_1 \geq j$, $j_1 \geq j_3 \geq j_2$, $j_3 + j \geq j_1 + j_2$.

Examined below will be the 6j-symbols corresponding to the recoupling of the Kronecker product of three UIR belonging to the discrete positive series $D^{j_1^+} \otimes D^{j_2^+} \otimes D^{j_3^+}$ of the SU(1, 1) group. Since these quantities differ from the T-coefficients in the tree-graph method for hyperspherical harmonics by a phase factor only (Smirnov and Shitikova 1977) it will be sufficient to consider the T-coefficients. Using the results of Kildyushov (1972), we may obtain

$$\begin{Vmatrix} j_1 & j_2 & j_3 \\ j_{12} & j & j_{23} \end{Vmatrix} = \frac{(-1)^{j_{12}-j_1-j_2-1}}{d_{j_{12}-j_1-j_2-1}} [\rho(j_{23})(2j_{23}+1)]^{1/2} \times v^{(2j_1+1, 2j_2+1)}_{j_{12}-j_1-j_2-1}[j_{23}(j_{23}+1); j_2+j_3+1, j-j_1], \tag{4.2}$$

where $\rho(x)$ and d_n are respectively the weight and norm of the Racah polynomials $v_n^{(\alpha,\beta)}(p, a, b)$.

Let us mention the asymptotic formulae of the second order of accuracy for the 6j-symbols. Using the asymptotic representation of the Racah polynomials (Nikiforov *et al* 1983b)

$$v_n^{(\alpha,\beta)}(p; a, b) = (\tilde{N}^2)^n [P_n^{(\alpha,\beta)}(s) + O(\tilde{N}^{-2})],$$
$$\tilde{N}^2 = (b + \alpha/2)^2 - (a - \beta/2)^2, \qquad b \to \infty, \tag{4.3}$$

where

$$p = x(x+1) = -\tfrac{1}{4} + (a - \beta/2)^2(1-s)/2 + (b + \alpha/2)^2(1+s)/2,$$

we find for the 6j-symbols of the SU(2) group (Nikiforov *et al* 1983b)

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} \approx \frac{(-1)^{j_2+j_3+j_{23}}}{[(j_1+j_2+1)(j_3+j+1)]^{1/2}} d^{j_{12}}_{j_1-j_2, j_3-j}(\beta). \tag{4.4}$$

Here (see figure 5)

$$\cos\beta = \frac{(2j_{23}+1)^2-(j_1+j_2+1)^2-(j_3+j+1)^2}{2(j_1+j_2+1)(j_3+j+1)}.$$

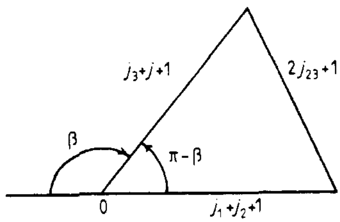

Figure 5.

In the case of the SU(1, 1) group, the analogous formula is of the form

$$\left\|\begin{matrix} j_1 & j_2 & j_3 \\ j_{12} & j & j_{23} \end{matrix}\right\| \approx (-1)^{j_{12}+j_1-j_2}\left(\frac{(2j_{12}+1)(2j_{23}+1)}{(j-j_3)(j+j_3+1)}\right)^{1/2} d^{j_{12}}_{j_1+j_2+1, j_2-j_1}(\beta), \tag{4.5}$$

$$j \gg j_1 \sim j_2 \sim j_{12} \sim j_3 \sim 1,$$

where (see figure 6)

$$\cos\beta = \frac{(2j_{23}+1)^2-(j-j_3)^2-(j+j_3+1)^2}{2(j-j_3)(j+j_3+1)}.$$

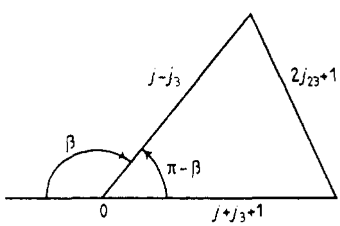

Figure 6.

The 6j-symbols of the SU(2) group are known to coincide with the transformation brackets between the T, V and U bases of the SU(3) group (see e.g. Pluhar *et al* 1981). This circumstance makes it possible to find the group theoretical properties of the Racah polynomials. It has appeared that the action of the operators $T_\pm$, $U_\pm$, and $V_\pm$ on the corresponding Gelfand–Tsetlin bases of the SU(3) group is equivalent to the calculation of the first derivative for the given polynomials. The operators T^2, U^2 and V^2 lead to a difference equation, to a three-term recurrence relation, etc.

The asymptotic properties of the 6j-symbols may also be analysed by turning from difference equations to differential ones, after the analogy of § 2. The same method may be used to examine the arbitrary three-term recurrence relations. To make the discussion more complete, we shall illustrate the method additionally by two physically interesting examples, namely the Hamiltonian of an asymmetric top and the operator Ω of Bargmann and Moshinsky (1961).

5. The asymptotic properties of the Hamiltonian of an asymmetric top and the Bargmann–Moshinsky operator Ω in the SU(3) scheme

The Hamiltonian of an asymmetric top is of the form (Bohr and Mottelson 1974)

$$H = A_1\mathcal{J}_1^2 + A_2\mathcal{J}_2^2 + A_3\mathcal{J}_3^2 \tag{5.1}$$

where $A_i = \hbar^2/2I_i$, $i = 1, 2, 3$ (I_i are the main momenta of inertia).

The matrix of this Hamiltonian in the eigenfunction basis of a symmetrical top

$$|\mathcal{J}MK\rangle = [(2\mathcal{J}+1)/8\pi^2]^{1/2} D^{\mathcal{J}*}_{MK}(\alpha, \beta, \gamma)$$

(α, β, γ are the Eulerian angles, K is the projection of the angular momentum onto axis 3) is three-diagonal:

$$\begin{aligned} &H_{KK} = B\mathcal{J}(\mathcal{J}+1) + D[3K^2 - \mathcal{J}(\mathcal{J}+1)],\\ &H_{K\pm2,K} = F[(\mathcal{J}\mp K)(\mathcal{J}\pm K+1)(\mathcal{J}\mp K-1)(\mathcal{J}\pm K+2)]^{1/2},\\ &B = \tfrac{1}{3}(A_1+A_2+A_3), \qquad D = \tfrac{1}{6}(2A_3 - A_1 - A_2), \qquad F = \tfrac{1}{4}(A_1 - A_2). \end{aligned} \tag{5.2}$$

As a result, for the coefficients C_K of the expansion of the wavefunction of an asymmetric top in eigenfunctions of a symmetric top

$$|\mathcal{J}M\varepsilon\rangle = \sum_{K=-\mathcal{J}}^{\mathcal{J}} C_K|\mathcal{J}MK\rangle \tag{5.3}$$

we obtain the three-term recurrence relation

$$H_{K+2,K}C_{K+2} + (H_{KK} - \varepsilon)C_K + H_{K-2,K}C_{K-2} = 0. \tag{5.4}$$

The asymptotic properties of the eigenvalues of energy ε and wavefunctions of an asymmetric top at $\mathcal{J} \gg 1$ may be obtained by turning from the difference equations (5.4) to a differential equation of second order, similarly to the procedure used for CGC.

It should be noted that the set of equations (5.4) is actually broken into two subsets with $K = \mathcal{J}-1, \mathcal{J}-3, \ldots$ and $K = \mathcal{J}, \mathcal{J}-2, \mathcal{J}-4, \ldots$ (for integral (half-integral) $\mathcal{J}$ the first subset contains $\mathcal{J}+1$ ($\mathcal{J}+\frac{1}{2}$) equations, and the second one contains $\mathcal{J}$ ($\mathcal{J}+\frac{1}{2}$) equations).

In the extreme case of large rotational angular momenta $\mathcal{J}$ the discrete variable K, which varies with step $\Delta K = 2$, will be treated as continuous. We shall be interested in the lowest levels with a given $\mathcal{J}$ whose wavefunctions in the K representation $C_K \approx C(K)$ reverse its sign a few times throughout the total interval of variations of the variable $-\mathcal{J} \leqslant K \leqslant \mathcal{J}$. In these states the wavefunction is mainly concentrated near small values of K (i.e. $|C_K| \ll 1$ at $|K| \sim \mathcal{J}$).

To turn from the difference equations (5.4) to a differential equation, we shall use the Taylor-series expansion of the coefficients $C_{K\pm2}$ near the point K in powers $1/\mathcal{J}$, and the Taylor-series expansion of the matrix elements $H_{K\pm2,K}$ in powers $1/n$ and

$1/(N-n)$ ($N=2\mathcal{J}$, $n=\mathcal{J}-K \gg 1$ and $N-n=\mathcal{J}+K \gg 1$) to within the terms of second order. After replacing the variable $K/\mathcal{J}=\sin\theta$, we get the following equation for the function $W(\theta)=(\cos\theta)^{1/2}C$:

$$-(A_2-A_1)\,\mathrm{d}^2W/\mathrm{d}\theta^2+[\mathcal{J}(\mathcal{J}+1)A_1+(A_3-A_1)\mathcal{J}^2\sin^2\theta]W=\varepsilon W \tag{5.5}$$

with boundary condition $W(\pm\pi/2)=0$. In the limiting case $\theta \ll 1$ equation (5.5) takes the form of the Schrödinger equation for a one-dimensional harmonic oscillator with the frequency

$$\hbar\omega=2\mathcal{J}[(A_3-A_1)(A_2-A_1)]^{1/2} \tag{5.6}$$

(the same result may be obtained by expanding the left-hand side of (5.4) in powers $1/\mathcal{J}$ to within the terms of third order). Therefore the spectrum of the lowest levels of an asymmetric top to a harmonic approximation is of the form

$$\varepsilon_n^{(0)}=A_1\mathcal{J}(\mathcal{J}+1)+(n+\tfrac{1}{2})\hbar\omega \tag{5.7a}$$

(we suppose $A_1<A_2<A_3$).

A similar expression was obtained, for example, by Bohr and Mottelson (1974) using the boson expansion.

Equation (5.5) permits one to make this result more accurate. The potential term in the Schrödinger equation (5.5) differs from a parabola and, therefore, the level spectrum in (5.5) is not equidistant. This deviation may be made allowance for by replacing the potential $V=b\sin^2\theta$ with the modified Pöschl–Teller potential $A(1-\cosh^{-2}(\alpha\theta))$ which coincides at $A=2b$ and $\alpha=1/\sqrt{2}$ with the potential V to within the terms proportional to θ^5. Since the level spectrum for the modified Pöschl–Teller potential is known (Flügge 1971), we obtain to this approximation

$$\varepsilon_n^{(1)}=\varepsilon_n^{(0)}+\Delta\varepsilon_n, \qquad \Delta\varepsilon_n=\tfrac{1}{2}(A_1-A_2)(n+\tfrac{1}{2})^2. \tag{5.8a}$$

By choosing the axis with the maximum moment of inertia (axis 1 in our case) to be the quantisation axis, we may obtain formulae analogous to (5.7a)–(5.8a) for the levels with the maximum energy at a given $\mathcal{J}$ by merely replacing $A_1 \leftrightarrow A_3$ and by changing sign of the second term in (5.7a) and (5.8a):

$$\varepsilon_n^{(0)}=A_3\mathcal{J}(\mathcal{J}+1)-(n+\tfrac{1}{2})2\mathcal{J}[(A_3-A_2)(A_3-A_1)]^{1/2}; \tag{5.7b}$$

$$\varepsilon_n^{(1)}=\varepsilon_n^{(0)}+\Delta\varepsilon_n, \qquad \Delta\varepsilon_n=\tfrac{1}{2}(A_3-A_2)(n+\tfrac{1}{2})^2. \tag{5.8b}$$

Table 1 presents the results of the calculations of the level energies of an asymmetric top at $\mathcal{J}=18$ and $A_1=\frac{1}{2}$, $A_2=1$, $A_3=\frac{3}{2}$. The values of $\varepsilon_n^{(0)}$ and $\varepsilon_n^{(1)}$ presented in table 1 have been calculated using (5.7a) and (5.8a) respectively for the lower part of the spectrum and (5.7b) and (5.8b) respectively for the upper part of the spectrum. The values of ε_n have been obtained by numerical diagonalisation of the matrix (5.2) (the band with $K=\mathcal{J}-1$, $\mathcal{J}-3,\ldots$). From table 1 one can easily see that the values of $\varepsilon_n^{(0)}$ and $\varepsilon_n^{(1)}$ are different from the exact energy values of five or six lowest or highest levels of an asymmetric top with $\mathcal{J}=18$ by not more than 5%. In our approach there is the degeneracy of the bands with $K=\mathcal{J}-1$, $\mathcal{J}-3,\ldots$ and $K=\mathcal{J}$, $\mathcal{J}-2$, $\mathcal{J}-4,\ldots$ which is confirmed by exact results to within a good accuracy for several highest and lowest levels.

Figure 7 shows several wavefunctions of an asymmetric top. It is seen that in the case of lowest (highest) levels they are actually concentrated in the region of small K,

Table 1. The level spectrum of an asymmetric top ($\mathcal{J}=18$, $A_1=\frac{1}{2}$, $A_2=1$, $A_3=\frac{3}{2}$; the band with $K=\mathcal{J}-1, \mathcal{J}-3, \ldots$).

$\varepsilon_n^{(0)}$	$\varepsilon_n^{(1)}$	ε_n	$\varepsilon_n^{(0)}$	$\varepsilon_n^{(1)}$	ε_n
183.8	183.6	183.7	343.3	368.7	376.1
210.0	208.1	208.3	369.4	388.1	392.7
236.1	231.5	231.4	395.6	408.6	411.0
262.3	254.0	253.1	421.7	430.0	431.0
288.4	275.4	273.0	447.9	452.5	452.5
314.6	295.9	291.3	474.0	475.6	475.6
340.8	315.3	307.8	500.2	500.4	500.3
		322.0			
		335.5			
...	...	348.2			
		361.7			

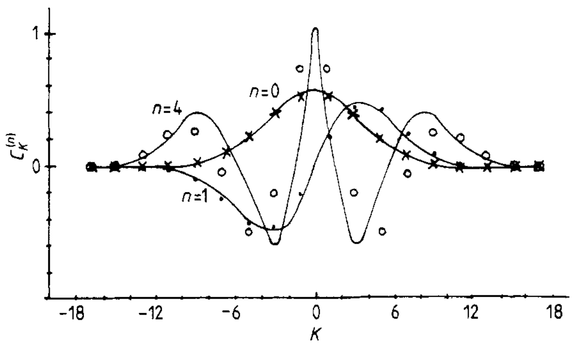

Figure 7. The exact values of the coefficients $C_K^{(n)}$ for the lowest levels of the asymmetric top ($\mathcal{J}=18$, $A_1=\frac{1}{2}$, $A_3=\frac{3}{2}$, $A_2=1$) with $n=0$ (×), $n=1$ (·) and $n=4$ (○). The asymptotical harmonic oscillator wavefunctions are represented by full curves. $K=\mathcal{J}-1, \mathcal{J}-3, \ldots$.

have a definite parity with respect to the replacement $K\to -K$ and their form is similar to the form of the lowest eigenfunctions of the harmonic oscillator.

Similarly to the above, we may analyse the asymptotic properties of the Bargmann–Moshinsky operator Ω (Bargmann and Moshinsky 1961) in the SU(3) scheme:

$$\Omega=([L\times L]^2\cdot Q) \tag{5.9}$$

which has been studied poorly up to now. In (5.9), L is the orbital angular momentum of the system; Q is the quadrupole momentum operator (Elliott 1958), the generator of the SU(3) group. We shall seek for its eigenfunctions Ψ_ω satisfying the condition

$$\Omega\Psi_\omega=\omega\Psi_\omega \tag{5.10}$$

in the form of expansion in the projected Elliott basis $|(\lambda\mu)LMK\rangle=P^L_{MK}\chi_K$ for the irreducible representation $(\lambda\mu)$ of the SU(3) group (P^L_{MK} is the projecting operator; χ_K is the 'internal' function of the system):

$$\Psi_\omega=\sum_K C_K|(\lambda\mu)LMK\rangle. \tag{5.11}$$

The following expression is valid (Filippov *et al* 1981):

$$\Omega|(\lambda\mu)LMK\rangle = \sum_{K'=K,K\pm 2} \Omega_{K'K}|(\lambda\mu)LMK'\rangle \tag{5.12}$$

where

$$\Omega_{KK} = -\tfrac{1}{2}[1+\tfrac{1}{3}(2\lambda+\mu)][L(L+1)-3K^2],$$
$$\Omega_{K\pm 2,K} = -\tfrac{1}{4}[(\mu\mp K)(\mu\pm K+2)(L\pm K+2)(L\mp K-1)(L\pm K+1)(L\mp K)]^{1/2}. \tag{5.13}$$

Therefore, we get the following difference equation for C_K:

$$\Omega_{K-2,K}C_{K-2}+(\Omega_{KK}-\omega)C_K+\Omega_{K+2,K}C_{K+2}=0. \tag{5.14}$$

The similarity of these relations to (5.3), (5.4) is obvious. The only difference is that K takes on only values $K=\mu, \mu-2, \ldots, -\mu$ with $|K|\leq L$. Besides that, since the relation

$$|(\lambda\mu)LM-K\rangle = (-1)^{L+\lambda}|(\lambda\mu)LMK\rangle \tag{5.15}$$

is valid for the Elliott basis, we should only be interested in such solutions of (5.14) which show a definite parity with respect to the replacement $K\to -K$, i.e. the condition

$$C_K = (-1)^{L+\lambda}C_{-K} \tag{5.16}$$

must be satisfied.

Examine now the case λ, μ, $L\gg 1$. We shall be interested in the lowest eigenvalues ω. In this case the coefficients C_K satisfying the conditions $K\ll L$ and $K\ll\mu$ are dominating in the corresponding wavefunctions. Then the left-hand side of the recurrence relation (5.14) may be expanded into power series with respect to K/L and K/μ retaining the terms of second order. As a result we get the Schrödinger equation with the potential of the harmonic oscillator whose solutions are

$$C_K^{(n)} = N\exp(-\xi^2K^2/2)H_n(\xi K),$$
$$N=\left(\frac{\xi}{2^{n-1}n!\sqrt{\pi}}\right)^{1/2}, \qquad \xi=\left(\frac{2+\lambda+\mu}{L(L+1)(\mu+1)}+\frac{\mu-1}{4\mu^2(\mu+1)}\right)^{1/4} \tag{5.17}$$

(where $H_n(x)$ is the Hermite polynomial) with eigenvalues

$$\omega_n^{(0)} = G_0+(n+\tfrac{1}{2})H_0,$$
$$G_0 = -L(L+1)[1+\tfrac{2}{3}\mu+\tfrac{1}{3}\lambda+(\mu+1)/2L(L+1)+(\mu-1)/4\mu^2],$$
$$H_0 = 2\{L(L+1)(\mu+1)[2+\lambda+\mu+L(L+1)(\mu-1)/4\mu^2]\}^{1/2}. \tag{5.18}$$

From the solutions, those showing the symmetry property (5.16) must be selected.

The given approximation describes, generally speaking, only the lowest part of the spectrum of the eigenvalues ω for the irreducible representation $(\lambda\mu)$ of the SU(3) group. The highest part of the spectrum for this representation may be obtained from the lowest part of the ω spectrum for the conjugated irreducible representation $(\mu\lambda)$ by the sign inversion

$$\omega(\lambda\mu) = -\omega(\mu\lambda). \tag{5.19}$$

The approximation (5.18) may be made more accurate by expanding (5.14) to within the terms of third order in powers K/L and K/μ. Besides that, at $\mu>L$ the matrix elements $\Omega_{K\pm 2,K}$ in (5.14) may be expanded in powers $1/(L+K)$, $1/(L-K)$, K/μ and

$1/\mu$. As a result, we get the Schrödinger equation with a potential of the form $V = p\sin^2\theta - q\sin^4\theta$ (the meaning of θ is the same as in (5.5)). By replacing the potential by the modified Pöschl–Teller potential

$$\frac{2p^2}{p+3q}\left\{1-\cosh^{-2}\left[\left(\frac{p+3q}{2p}\right)^{1/2}\theta\right]\right\}$$

which coincides with V to within the terms proportional to θ^5, we may also obtain the spectrum of ω which differs from the equidistant spectrum and is more accurate than (5.18):

$$\begin{aligned}
&\omega_n^{(1)} = G_1 + (n+\tfrac{1}{2})H_1 + \Delta\omega_n,\\
&G_1 = -L(L+1)(1+\tfrac{2}{3}\mu+\tfrac{1}{3}\lambda) + L^2/4\mu, \qquad H_1 = 2(\mu+1)\sqrt{p},\\
&\Delta\omega_n = -(n+\tfrac{1}{2})^2(\mu+1)(p+3q)/2p,\\
&p = \frac{L^2}{\mu+1}\left(2+\lambda+\mu+\frac{L^2-1}{4\mu}\right), \qquad q = \frac{L^4}{4\mu(\mu+1)}.
\end{aligned} \tag{5.20}$$

Presented in table 2 are the results of calculating the eigenvalues of ω for $(\lambda\mu) = (22, 18)$ and $L = 18$. The values of ω_n have been obtained by numerical diagonalisation of the matrix (5.13). For the lower part of the spectrum the values of $\omega_n^{(0)}$ and $\omega_n^{(1)}$ have been calculated by (5.18) and (5.20) respectively; for the upper part of the spectrum the values of $\omega_n^{(0)}$ and $\omega_n^{(1)}$ have been obtained by the sign inversion (5.19) of the eigenvalues of Ω calculated by (5.18) and (5.20) for the conjugated irreducible representation $(\lambda\mu) = (18, 22)$.

Table 2. Eigenvalues ω of the Bargmann–Moshinsky operator Ω ($(\lambda\mu) = (22, 18)$, $L = 18$).

$\omega_n^{(0)}$	$\omega_n^{(1)}$	ω_n
−6418	−6418	−6424
−4220	−4351	−4458
−2022	−2383	−2801
		−1466
		−578.6
...	...	+111.0
		1292
2029	2445	2823
4427	4579	4670
6827	6827	6830

Obviously the accuracy of (5.18) and (5.20) is not high, but these formulae give us the correct impression concerning the structure of the spectrum of Ω.

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Appendix 1. The connection of D-functions for the SU(1, 1) group with Meixner polynomials

The generalised spherical harmonics of the SU(2) group may be expressed through the Kravchuk polynomials (Koornwinder 1982). Therefore, the unitarity property of the d-functions of the SU(2) group is equivalent to the orthogonality property of the Kravchuk polynomials on a discrete set of points.

The generalised spherical harmonics (the Bargmann functions) in the case of the discrete positive series D^{j+} of the SU(1, 1) group (Barut and Wilson 1976)

$$V^j_{nn'}(\beta)=\frac{(-1)^{n'-j-1}}{(2j+1)!}\left(\frac{(n+j)!(n'+j)!}{(n-j-1)!(n'-j-1)!}\right)^{1/2}\left(\tanh\frac{\beta}{2}\right)^{n+n'}$$
$$\times\left(\sinh\frac{\beta}{2}\right)^{-2j-2}F\left(-n+j+1,-n'+j+1,2j+2,-\frac{1}{\sinh^2(\beta/2)}\right) \quad \text{(A1.1)}$$

are closely related to the Meixner polynomials $m_n^{(\gamma)}(x,\mu)$:

$$V^j_{nn'}(\beta)=(-1)^{n'-j-1}\left(\frac{(n'+j)!}{(n+j)!(n-j-1)!(n'-j-1)!}\right)^{1/2}\left(\tanh\frac{\beta}{2}\right)^{n+n'}$$
$$\times\left(\sinh\frac{\beta}{2}\right)^{-2j-2}m^{(2j+2)}_{n-j-1}\left(n'-j-1,\tanh^2\frac{\beta}{2}\right). \quad \text{(A1.2)}$$

In this case the unitarity of the Bargmann functions

$$\sum_{n'=j+1}^{+\infty} V^j_{nn'}(\beta)\,V^j_{n''n'}(\beta)=\delta_{nn''} \quad \text{(A1.3)}$$

is equivalent to the orthogonality property of the Meixner polynomials

$$\sum_{k=0}^{+\infty} m_n^{(\gamma)}(k,\mu)m_{n'}^{(\gamma)}(k,\mu)\rho(k)=d_n^2\delta_{nn'} \quad \text{(A1.4)}$$

where

$$\rho(x)=\mu^x\Gamma(\gamma+x)/\Gamma(x+1)\Gamma(\gamma), \qquad d_n^2=n!(\gamma)_n/\mu^n(1-\mu)^\gamma.$$

Appendix 2. The standardisation of the Racah and Hahn polynomials

For information, we shall mention some properties of the Racah and Hahn polynomials (Karlin and McGregor 1961, Nikiforov *et al* 1982, 1983b).

The Racah polynomials and the dual Hahn polynomials are orthogonal on a quadratic grid

$$\sum_{i=a}^{b-1} y_n(p_i)y_{n'}(p_i)\rho_i\Delta p_{i-1/2}=d_n^2\delta_{nn'}, \quad \text{(A2.1)}$$

where

$$p_i=i(i+1), \qquad \Delta p_{i-1/2}=2i+1, \qquad \rho_i=\rho(i).$$

The values of the weight $\rho(x)$ and of the squared norm d_n^2 are presented in table 3 which shows the values of the first coefficient a_n in the explicit form of these polynomials

$$y_n(p)=a_np^n+b_np^{n-1}+\ldots. \quad \text{(A2.2)}$$

Table 3. Standardisation of the Racah polynomials $v_n^{(\alpha,\beta)}(p, a, b)$ and Hahn polynomials $h_n^{(\alpha,\beta)}(x, N)$, $w_n^{(c)}(p, a, b)$.

$y_n(p)$	$v_n^{(\alpha,\beta)}(p, a, b), p = x(x+1)$	$h_n^{(\alpha,\beta)}(x, N)$	$w_n^{(c)}(p, a, b), p = x(x+1)$
(a, b)	(a, b)	$(0, N)$	(a, b)
$\rho(x)$	$\frac{\Gamma(a+x+1)\Gamma(x-a+\beta+1)}{\Gamma(x-a+1)\Gamma(a-\beta+x+1)} \times \frac{\Gamma(b+\alpha-x)\Gamma(b+\alpha+x+1)}{\Gamma(b-x)\Gamma(b+x+1)}$ $(-\frac{1}{2} < a \leqslant b-1;\ \alpha > -1;\ -1 < \beta < 2a+1)$	$\frac{\Gamma(\alpha+N-x)\Gamma(\beta+x+1)}{\Gamma(N-x)\Gamma(x+1)}$ $(\alpha, \beta > -1)$	$\frac{\Gamma(a+x+1)\Gamma(c+x+1)}{\Gamma(x-a+1)\Gamma(x-c+1)} \times [\Gamma(b-x)\Gamma(b+x+1)]^{-1}$ $(-\frac{1}{2} < a \leqslant b-1;\ \lvert c\rvert < a+1)$
a_n	$(1/n!)(\alpha+\beta+n+1)_n$	$(1/n!)(\alpha+\beta+n+1)_n$	$1/n!$
d_n^2	$\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)} \times \frac{(b-a+\alpha+\beta+n+1)\Gamma(a+b+\alpha+n+1)}{(b-a-n-1)!\Gamma(a+b-\beta-n)}$	$\frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{(\alpha+\beta+2n+1)n!\Gamma(\alpha+\beta+n+1)} \times \frac{\Gamma(\alpha+\beta+n+N+1)}{(N-n-1)!}$	$\frac{\Gamma(a+c+n+1)}{n!(b-a-n-1)!\Gamma(b-c-n)}$

By setting the values $\rho(x)$ and a_n, we determine the polynomials $v_n^{(\alpha,\beta)}(p)$ and $w_n(p)$ unambiguously.

The Hahn polynomials are orthogonal on a linear grid

$$\sum_{i=a}^{b-1} y_n(i) y_{n'}(i) \rho(i) = d_n^2 \delta_{nn'}. \tag{A2.3}$$

The values of $\rho(x)$, d_n^2 and a_n for these polynomials are also presented in table 3.

The Hahn polynomials $h_n^{(\alpha,\beta)}(x, N)$ are related to the dual Hahn polynomials $w_n^{(c)}(p; a, b)$, $p = x(x+1)$ as (Karlin and McGregor 1961)

$$h_n^{(\alpha,\beta)}(k, N) = (-1)^{k-n} \frac{k!(N-k-1)\Gamma(\beta+n+1)}{n!(N-n-1)!\Gamma(\beta+k+1)} \times w_k^{((\beta-\alpha)/2)}\left[p\left(\frac{\alpha+\beta}{2}+n\right); \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}+N\right]$$
$$(k, n = 0, 1, \ldots, N-1). \tag{A2.4}$$

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